

ON THE NUMBER OF COMMUTATION CLASSES OF THE LONGEST ELEMENT IN THE SYMMETRIC GROUP

HUGH DENONCOURT, DANA C. ERNST, AND DUSTIN STORY

ABSTRACT. Using the standard Coxeter presentation for the symmetric group S_n , two reduced expressions for the same group element are said to be commutation equivalent if we can obtain one expression from the other by applying a finite sequence of commutations. The resulting equivalence classes of reduced expressions are called commutation classes. How many commutation classes are there for the longest element in S_n ?

Original proposer of the open problem: Donald E. Knuth

The year when the open problem was proposed: 1992 [11, §9]

A *Coxeter system* is a pair (W, S) consisting of a distinguished (finite) set S of generating involutions and a group

$$W = \langle S \mid (st)^{m(s,t)} = e \text{ for } m(s,t) < \infty \rangle,$$

called a *Coxeter group*, where e is the identity, $m(s,t) = 1$ if and only if $s = t$, and $m(s,t) = m(t,s)$. It turns out that the elements of S are distinct as group elements and that $m(s,t)$ is the order of st . Since the elements of S have order two, the relation $(st)^{m(s,t)} = e$ can be written to allow the replacement

$$\underbrace{sts \cdots}_{m(s,t)} \mapsto \underbrace{tst \cdots}_{m(s,t)}$$

which is called a *commutation* if $m(s,t) = 2$ and a *braid move* if $m(s,t) \geq 3$.

Given a Coxeter system (W, S) , a word $\mathbf{w} = s_{x_1}s_{x_2} \cdots s_{x_m}$ in the free monoid S^* is called an *expression* for $w \in W$ if it is equal to w when considered as a group element. If m is minimal among all expressions for w , the corresponding word is called a *reduced expression* for w . In this case, we define the *length* of w to be $\ell(w) = m$. According to [8], every finite Coxeter group contains a unique element of maximal length, which we refer to as the *longest element* and denote by w_0 .

Let (W, S) be a Coxeter system and let $w \in W$. Then w may have several different reduced expressions that represent it. However, Matsumoto's Theorem [7, Theorem 1.2.2]

| | | | | | | | |
|--------|--------|--------|--------|--------|--------|--------|--------|
| | 312312 | | | | 231231 | | |
| 321323 | 132312 | | | 123121 | 213231 | | |
| 323123 | 312132 | 321232 | 232123 | 121321 | 231213 | 123212 | 212321 |
| | 132132 | | | | 213213 | | |

FIGURE 1. Reduced expressions and the corresponding commutation classes for the longest element in S_4 .

states that every reduced expression for w can be obtained from any other by applying a finite sequence of commutations and braid moves.

Following [13], we define a relation \sim on the set of reduced expressions for w . Let \mathbf{w} and \mathbf{w}' be two reduced expressions for w and define $\mathbf{w} \sim \mathbf{w}'$ if we can obtain \mathbf{w}' from \mathbf{w} by applying a single commutation. Now, define the equivalence relation \approx by taking the reflexive transitive closure of \sim . Each equivalence class under \approx is called a *commutation class*.

The Coxeter system of type A_{n-1} is generated by $S(A_{n-1}) = \{s_1, s_2, \dots, s_{n-1}\}$ and has defining relations (i) $s_i s_i = e$ for all i ; (ii) $s_i s_j = s_j s_i$ when $|i - j| > 1$; and (iii) $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$. The corresponding Coxeter group $W(A_{n-1})$ is isomorphic to the symmetric group S_n under the correspondence $s_i \mapsto (i, i + 1)$. It is well known that the longest element in S_n is given in 1-line notation by

$$w_0 = [n, n - 1, \dots, 2, 1]$$

and that $\ell(w_0) = \binom{n}{2}$.

Let c_n denote the number of commutation classes of the longest element in S_n . The longest element w_0 in S_4 has length 6 and is given by the permutation $(1, 4)(2, 3)$. There are 16 distinct reduced expressions for w_0 while $c_4 = 8$. The 8 commutation classes for w_0 are given in Figure 1, where we have listed the reduced expressions that each class contains. Note that for brevity, we have written i in place of s_i .

In [12], Stanley provides a formula for the number of reduced expressions of the longest element w_0 in S_n . However, the following question is currently unanswered.

Open Problem. *What is the number of commutation classes of the longest element in S_n ?*

To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [11], but not using our current terminology. A more general version of the problem appears in Section 5.2 of [9]. In the paragraph following the proof of Proposition 4.4 of [14], Tenner explicitly states the open problem in terms of commutation classes.

According to sequence A006245 of The On-Line Encyclopedia of Integer Sequences [1], the first 10 values for c_n are 1, 1, 2, 8, 62, 908, 24698, 1232944, 112018190, 18410581880. To

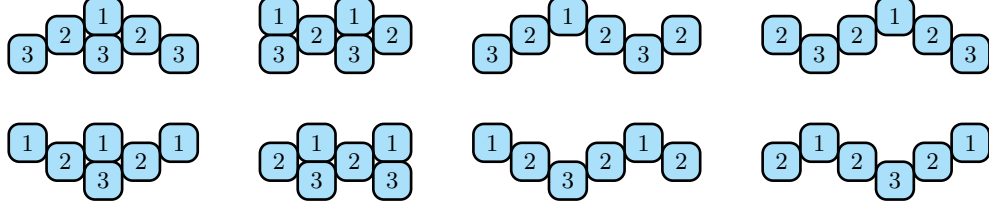
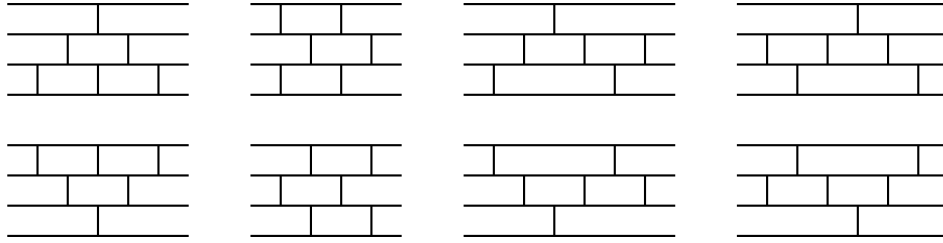

 FIGURE 2. Heaps for the longest element in S_4 .


FIGURE 3. Minimal ladder lotteries corresponding to the primitive sorting networks on 4 elements.

date, only the first 15 terms are known. The current best upper-bound for c_n was obtained by Felsner and Valtr. They prove that for sufficiently large n , $c_n \leq 2^{0.6571n^2}$ [5, Theorem 2], although their result is stated in terms of arrangements of pseudolines.

The commutation classes of the longest element of the symmetric group are in bijection with a number of interesting objects. It turns out that c_n is equal to the number of

- heaps for the longest element in S_n [13, Proposition 2.2];
- primitive sorting networks on n elements [2, 10, 11, 15, 16];
- rhombic tilings of a regular $2n$ -gon (where all side lengths of the rhombi and the $2n$ -gon are the same) [3, 14];
- oriented matroids of rank 3 on n elements [6, 9];
- arrangements of n pseudolines [4, 5, 11].

In Figure 2, we have drawn lattice point representations of the 8 heaps that correspond to the commutation classes for the longest element in S_4 . Note that our heaps are sideways versions of the heaps that usually appear in the literature. The minimum ladder lotteries (or ghost legs) corresponding to the 8 primitive sorting networks on 4 elements are provided in Figure 3. The 8 distinct rhombic tilings of a regular octagon are depicted in Figure 4.

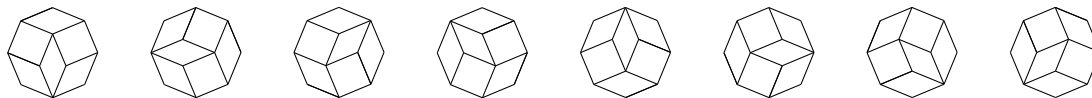


FIGURE 4. Rhombic tilings of a regular octagon.

Very little is known about the number of commutation classes of the longest element in other finite Coxeter groups.

REFERENCES

- [1] The On-Line Encyclopedia of Integer Sequences (OEIS). <http://oeis.org>, 2016.
- [2] D. Armstrong. The sorting order on a Coxeter group. *J. Combin. Theory, Ser. A*, 2009.
- [3] S. Elnitsky. Rhombic Tilings of Polygons and Classes of Reduced Words in Coxeter Groups. *J. Combin. Theory, Ser. A*, 77(2), 1997.
- [4] S. Felsner. On the Number of Arrangements of Pseudolines. *Discrete Comput. Geom.*, 18(3), 1997.
- [5] S. Felsner and P. Valtr. Coding and Counting Arrangements of Pseudolines. *Discrete Comput. Geom.*, 46(3), 2011.
- [6] J. Folkman and J. Lawrence. Oriented matroids. *J. Combin. Theory, Ser. B*, 25(2), 1978.
- [7] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori–Hecke algebras*. 2000.
- [8] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, 1990.
- [9] M.M. Kapranov and V.A. Voevodsky. Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders (list of results). *Cahiers de topologie et géométrie différentielle catégoriques*, 32(1), 1991.
- [10] J. Kawahara, T. Saitoh, R. Yoshinaka, and S. Minato. Counting Primitive Sorting Networks by π DDs. *TCS Technical Report*, 2011.
- [11] D. Knuth. *Axioms and Hulls*. Springer-Verlag, Berlin, 1992.
- [12] R.P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5(4), 1984.
- [13] J.R. Stembridge. On the fully commutative elements of Coxeter groups. *J. Algebraic Combin.*, 5, 1996.
- [14] B.E. Tenner. Reduced decompositions and permutation patterns. *J. Algebraic Combin.*, 24(3), 2006.
- [15] K. Yamanaka and S. Nakano. Efficient Enumeration of All Ladder Lotteries with k Bars. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, E97-A(6), 2009.
- [16] K. Yamanaka, S. Nakano, Y. Matsui, R. Uehara, and K. Nakada. Efficient Enumeration of All Ladder Lotteries and Its Application. *Theoretical Computer Science*, 411, 2010.

THE CITY COLLEGE OF NEW YORK

E-mail address: `hdenoncourt@ccny.cuny.edu`

NORTHERN ARIZONA UNIVERSITY

E-mail address: `dana.ernst@nau.edu`, `dustin@nau.edu`